JOURNAL OF APPROXIMATION THEORY 20, 153–161 (1977)

H-Sets in Linear Approximation

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Communicated by Oved Shisha

Received May 16, 1974

This paper studies the role *H*-sets play in finding the best linear Tchebycheff approximation to a given continuous function. A simple definition is given for *H*-sets and the algebraic theory for linear approximation is developed. We find that many of the theorems where the Haar condition is supposed can be generalized in terms of *H*-sets; thus a general framework for Linear Tchebycheff Approximation is made.

1. NOTATION AND DEFINITIONS

We consider a compact topological space B and denote by $C(B)$ the set of continuous real- or complex-valued functions defined on B. Functions in $C(B)$ will be approximated in the Tchebycheff sense by the linear subspace V of $C(B)$ with a basis $\{g_1, g_2, ..., g_n\}$ and the degree of approximation to f by V will be denoted by $\rho(f)$. For a given f in $C(B)$ and any h in V, elements x of B satisfying $\|f - h\| = |f(x) - h(x)|$ will be referred to as norm points of h with respect to f; if h is a best Tchebycheff approximation (BTA) to f by V we will merely refer to them as norm points.

DEFINITION 1. The finite subset $\{x_1, x_2, ..., x_k\}$ of B is an *H-set* if and only if the matrix equation

$$\begin{pmatrix} g_1(x_1) & \cdots & g_1(x_k) \\ \vdots & & \vdots \\ g_n(x_1) & \cdots & g_n(x_k) \end{pmatrix} \begin{pmatrix} l_1 \\ \vdots \\ l_k \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

has a solution $l = (l_1, l_2, ..., l_k)$ with all $l_j \neq 0$. We then write the *H*-set as $[x_i, \lambda_i, e_i, k]$ where $\lambda_i = |l_i|$, $e_i = \operatorname{sgn} l_i$. We note that this definition leads directly to the Haar condition when V is a Tchebycheff system.

* Most of this work was done while studying for the degree of Ph.D. at the University of Lancaster, England, under the guidance of Professor A. Talbot, to whom I express grateful thanks. I also wish to thank the referee for his most helpful suggestions.

ISSN 0021-9045

We will refer to the H-set $[x_i, \lambda_i, e_i, k]$ and its point set $\{x_i\}$ by the same letter M and to the matrix relation defining the H-set as $A(M)\,\mathbf{l} = \theta$, where θ is the null vector.

DEFINITION 2. An H-set $[x_i, \lambda_i, e_i, k]$ is *minimal* when no subset of the point set $\{x_i\}$ forms an H-set.

We see immediately that every set of more than n points of B must contain a minimal H-set; also a maximal minimal H-set has exactly $(n + 1)$ points.

THEOREM 1.1. *A set* $M = \{x_1, x_2, ..., x_k\} \subseteq B$ *is an* H*-set with respect to* V *if and only if there exists a set of nonzero scalars* $\{l_i : i = 1, 2, ..., k\}$ *such that for every* h *in* V

$$\sum_{i=1}^{k} l_i h(x_i) = 0.$$

Proof. Every h in V can be written as $\boldsymbol{\alpha} \cdot \mathbf{g}'$ (prime denotes transpose), with $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $\mathbf{g} = (g_1, g_2, ..., g_n)$. The theorem follows when we consider the relationship

$$\boldsymbol{\alpha} A(M)\,\mathbf{l} = 0.$$

This theorem shows the equivalence of our definition of H-set and that of extremal signature [5].

Remark 1. If s is any nonzero scalar then the quadruple $[x_i, |s|\lambda_i, e_i \operatorname{sgn} s, k]$ is an H-set if and only if $[x_i, \lambda_i, e_i, k]$ is. In particular, by taking $s^2 = -1$, the real (Re) and imaginary (Im) parts of e_i are essentially interchanged.

LEMMA 1.1. *The zero vector* θ *is an element of the closed convex set* S*, a subset of the locally convex topological* T_1*-space* X*, if and only if there is no continuous linear functional* L *defined on* X *such that* Re $[L(x)] > 0$ *for all* $x \in S$.

Proof. Suppose $\theta \in S$; then there exist scalars λ_i and elements $x_i \in S$ such that

$$\theta = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_r x_r .$$

Then for every linear functional L on X we have

$$\sum_{i=1}^{r} \lambda_i L(x_i) = 0$$

which cannot be valid for an L for which Re $[L(x)] > 0$ for all $x \in S$.

Conversely, if θ does not belong to S; then the desired conclusion follows from separation theorems for convex sets.

THEOREM 1.2. *If* $[x_i, \lambda_i, e_i, k]$ *is an H-set with respect to V then there is no function* $h \in V$ *such that* $\mathrm{Re}[e_i h(x_i)] \geq 0$, $[\mathrm{Im}[e_i h(x_i)] \geq 0]$ for $i = 1, 2,\ldots, k$, *with strict inequality for some i. Conversely, if there is no* $h \in V$ *such that* $\mathrm{Re}[e_i h(x_i)] > 0$, $[\mathrm{Im}[e_i h(x_i)] > 0]$ *for* $i = 1, 2,\ldots, k$, *then* $[x_i, \lambda_i, e_i, k]$ *contains an H-set.*

Proof. Suppose $[x_i, \lambda_i, e_i, k]$ is an H-set. Then, from Theorem 1.1, there can be no $h \in V$ such that $\mathrm{Re}[e_i h(x_i)] \geq 0$ for $i = 1, 2,\ldots, k$, with strict inequality for some i.

For the converse, let $h = \boldsymbol{\alpha} \cdot \mathbf{g}'$ and $\mathbf{c}_i = (e_i g_1(x_i), e_i g_2(x_i),\ldots, e_i g_n(x_i))$. Then $h(x_i) = \boldsymbol{\alpha} \cdot \mathbf{c}_i'$. From the definition of linear functionals on R^n (or C^n) and the supposition, it follows that there can be no linear functional L defined on R^n (or C^n) such that $\mathrm{Re}[L(\mathbf{c}_i')] > 0$ for $i = 1, 2,\ldots, k$. Thus from Lemma 1.1, the zero vector θ must belong to the convex hull of the set of vectors $\{\mathbf{c}_i'\}$. Hence, there must exist a set of scalars, $\{\lambda_i\}$, real and non-negative, such that

$$\begin{pmatrix} g_1(x_1) & \cdots & g_1(x_k) \\ \vdots & & \vdots \\ g_n(x_1) & \cdots & g_n(x_k) \end{pmatrix} \begin{pmatrix} e_1\lambda_1 \\ \vdots \\ e_k\lambda_k \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix},$$

and the result follows.

From Remark 1, the theorem follows for $\mathrm{Im}[e_i h(x_i)]$. This theorem shows that there is a relationship between our definition of H-set and that of Collatz [1, 2, 3].

2. BASIC THEOREMS

From our definition of H-sets we can see that they must have many of the properties of alternants for Tchebycheff systems. We now state without proof two general theorems, the first can be found in [1, 2, 3], and the second in [4].

THEOREM 2.1. *Suppose on the point set* $\{x_i\}$ *of the H-set* $[x_i, \lambda_i, e_i, k]$, *the error in approximating the function f by* $h_0 \in V$ *is* $R_i = f(x_i) - h_0(x_i)$ *and* $\mathrm{sgn}\, R_i = \bar{e}_i$. *Then there is no* $h \in V$ *such that*

$$\mathrm{Re}[\bar{R}_i h(x_i)] > 0$$

or

$$\mathrm{Im}[\bar{R}_i h(x_i)] > 0 \qquad \textit{for all } i.$$

and we have

$$\inf_i \|R_i\| \le \rho(f) \le \|f - h_0\|.$$

THEOREM 2.2. *A function h_0 is a* BTA *to f by V if and only if the inequality*

$$\min_{x\in D} \mathrm{Re}[(\overline{f(x) - h_0(x)})\, h(x)] \le 0$$

holds for every $h \in V$, where D is the set of norm points of h_0 with respect to f.

These theorems lead us directly to the following important theorem which was first stated in [5]. We give here a more concise proof and we will refer to the theorem as the "Approximation Theorem."

THEOREM 2.3 (Approximation Theorem). *Given a function $f \in C(B)$ for which we want to find a* BTA *with respect to V. If a function $h_0 \in V$ can be found such that a subset of the norm points of h_0 with respect to f, viz., $x_1, x_2, \ldots, x_k$, is the point set of an H-set $[x_i, \lambda_i, e_i, k]$ and the error $R = f - h_0$ satisfies*

$$\mathrm{sgn}\, R(x_i) = \bar{e}_i, \quad i = 1, 2, \ldots, k,$$

then h_0 is a BTA *to f by V. Conversely, if h_0 is a* BTA *to f by V, then some finite subset of the norm points, say $\{x_1, x_2, \ldots, x_k\}$, and the scalar values*

$$e_i = \mathrm{sgn}(\overline{f(x_i) - h_0(x_i)}), \quad i = 1, 2, \ldots, k,$$

define an H-set $[x_i, \lambda_i, e_i, k]$.

Proof. For the first part we use Theorem 2.1, and get

$$\inf_i \|R_i\| = \|R\|;$$

hence the result. For the converse we use Theorem 2.2 according to which there is no n-tuple $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ such that

$$\mathrm{Re}[\mathbf{b} \cdot \mathbf{g}(x)(\overline{f(x) - h_0(x)})] > 0 \tag{1}$$

for all the norm points. Clearly the set of n-tuples $(g_1(x)\overline{(f(x) - h_0(x))}, \ldots, g_n(x)\overline{(f(x) - h_0(x))})$, where x ranges over the set of norm points, is compact. Thus from Lemma 1.1 and inequality (1) above, the zero vector θ must belong to the convex hull of this set of n-tuples. Hence there exist positive $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that

$$\sum_{i=1}^{k} \lambda_i g_j(x_i)(\overline{f(x_i) - h_0(x_i)}) = 0$$

for $j = 1, 2, \ldots, n$, the x_i being norm points. Setting $e_i = \operatorname{sgn}(f(x_i) - h_0(x_i))$, we have

$$\|f - h_0\| \sum_{i=1}^{k} \lambda_i e_i g_j(x_i) = 0$$

for $j = 1, 2, \ldots, n$, which defines an H-set $[x_i, \lambda_i, e_i, k]$.

It is a direct consequence of Theorem 2.3 that if h_0 is a BTA to f by V, then h_0 is also a BTA to f by V on every H-set chosen from the set of norm points. We now go further and show that the norm points of a BTA to f by V contain a subset common to every set of norm points of a BTA to f by V.

THEOREM 2.4. *If M is an H-set $[x_i, \lambda_i, e_i, k]$ contained in the set of norm points of h_1, a* BTA *to f by V, then it is also an H-set contained in the set of norm points of h_2, where h_2 is a* BTA *to f by V.*

Proof. We will assume that h_1 and h_2 are distinct functions of V. Because h_2, is a BTA to f by V, we have for $i = 1, 2, \ldots, k$,

$$\operatorname{Re}[e_i(f(x_i) - h_2(x_i))] \leqslant \rho(f) = e_i(f(x_i) - h_1(x_i)).$$

Thus

$$\operatorname{Re}[e_i(h_1(x_i) - h_2(x_i))] \leqslant 0 \qquad \text{for all } i.$$

If M is not contained in the set of norm points of h_2, then there must be at least one point giving strict inequality. Hence from Theorem 1.2, the quadruple $[x_i, \lambda_i, e_i, k]$ is not an H-set, a contradiction. Similarly, if the $\operatorname{sgn}(f(x_i) - h_2(x_i))$ do not equal the e_i, we get the same contradiction. Hence $[x_i, \lambda_i, e_i, k]$ must be an H-set contained in the set of norm points of h_2. We will denote by $N(f)$ this common set of points which are norm points of every BTA to f by V.

DEFINITION 3. The contour defined by a function h in V is

$$O(h) = \{x : x \in B,\ h(x) = 0\}.$$

THEOREM 2.5. *If there is no unique* BTA *to f by V, then the set $N(f)$ lies in some contour $O(h)$.*

Proof. Consider an H-set $M = [x_i, \lambda_i, e_i, k]$ contained in $N(f)$ and two different best approximations h_1 and h_2. From Theorem 2.4,

$$h_1(x_i) - h_2(x_i) = 0 \qquad \text{for } i = 1, 2, \ldots, k.$$

Hence the points $\{x_i\}$ lie in the contour $O(h_1 - h_2)$ and therefore so does the set $N(f)$. The theorem follows.

THEOREM 2.6. *An H-set $M = [x_i, \lambda_i, e_i, k]$ lies in a contour $O(h)$ if and only if the matrix $A(M)$ has rank less than n.*

Proof. We consider the set of homogenous equations

$$(\alpha_1, \alpha_2, \ldots, \alpha_n)\, A(M) = \theta.$$

For every nonzero $\boldsymbol{\alpha}$, these equations can be written as

$$h(x_i) = 0, \qquad i = 1, 2, \ldots, k,$$

where $h = \boldsymbol{\alpha}\mathbf{g}'$; equivalently, the points $\{x_i\}$ lie in the contour $O(h)$. However, there exists such a nonzero vector $\boldsymbol{\alpha}$ if and only if rank $A(M)$ is less than n.

We see from this theorem and Theorem 2.5 that if there is no unique BTA to f by V, then $N(f)$ cannot contain a maximal minimal H-set.

We now consider the dual problem and the standard theorem [4].

THEOREM 2.7. *For each $f \in C(B)$ there exists a linear functional L defined on $C(B)$ such that $L(f) = \rho(f)$, $L(h) = 0$ for all $h \in V$ and $\| L \| \leqslant 1$.*

Choose an H-set $[x_i, \lambda_i, e_i, k]$ and define the linear functional L associated with it as

$$L(f) = \sum_{i=1}^{k} \lambda_i e_i f(x_i).$$

Then $L(h) = 0$ for all $h \in V$ and

$$\| L \| \leqslant \sum_{i=1}^{k} \lambda_i ,$$

which sum we make unity.

THEOREM 2.8. *The maximal linear functional L as in Theorem 2.7 can be constructed from an H-set in the above manner.*

Proof. From the Approximation Theorem we know that some subset of the norm points defines an H-set $[x_i, \lambda_i, e_i, k]$. Then

$$L(f) = \sum \lambda_i e_i f(x_i).$$

For any algorithmic procedure we need to be able to construct a function h in V corresponding to some given H-set and to a function f in $C(B)$. For this we consider a set of k equations in $(n+1)$ unknowns $a_1, a_2, \ldots, a_n$ and α

$$\sum_{j=1}^{n} a_j g_j(x_i) + \alpha \bar{e}_i = f(x_i), \qquad i = 1, 2, \ldots, k$$

where the H-set is $[x_i, \lambda_i, e_i, k]$. We refer to these as the basic equations for $[x_i, \lambda_i, e_i, k]$. Observe that a useful specialization is to take, B as $\{x_i\}$.

THEOREM 2.9. *A function h defined by the basic equations for $[x_i, \lambda_i, e_i, k]$ is a* BTA *to f by V on the set* $\{x_i : i = 1, 2,..., k\}$.

Proof. Consider the basic equations

$$h(x_i) + \alpha \bar{e}_i = f(x_i), \qquad i = 1, 2,..., k:$$

the error at each point x_i is α, hence every x_i is a norm point. Also, sgn $[f(x_i) - h(x_i)] = \bar{e}_i$. Since $[x_i, \lambda_i, e_i, k]$ is an H-set, from the Approximation Theorem h is a BTA to f by V on the set $\{x_i : i = 1, 2,..., k\}$.

THEOREM 2.10. *The functions h satisfying the basic equations for an H-set $M = [x_i, \lambda_i, e_i, k]$ form a subset of V which is a translation of a subspace of V of dimension $n - r$, where r is the rank of $A(M)$.*

Proof. The basic equations for $[x_i, \lambda_i, e_i, k]$ can be written in the form

$$(a_1, a_2,..., a_n, \alpha) \cdot \begin{pmatrix} g_1(x_1) & \cdots & g_1(x_k) \\ & \vdots & \\ g_n(x_1) & \cdots & g_n(x_k) \\ \bar{e}_1 & \cdots & \bar{e}_k \end{pmatrix} = (f(x_1),..., f(x_k)).$$

The last row of the matrix must be linearly independent of the other n rows. The theorem follows.

We can now consider the set of BTA to f by V.

THEOREM 2.11. *The set of* BTA *to f by V is given by $h_1 - Q$, where $h_1 \in V$ is defined by basic equations and Q is a convex subset of V containing the zero function.*

Proof. Every H-set defines a basic set of equations. The solution to any such set say $(a_1, a_2,..., a_n, \alpha)$, will have the same value α in the $(n+1)$st position and in the case where the H-set is a subset of $N(f)$, $\alpha = \rho(f) = L(f)$. Conversely, every function h such that for a fixed value α,

$$f(x_i) - h(x_i) = \alpha \bar{e}_i \qquad \text{for all } i = 1, 2,..., k,$$

must satisfy the basic equations for $[x_i, \lambda_i, e_i, k]$.

From this we see that for any H-set the solution set for the basic equations is that set of functions which has a constant value for the difference $[f(x_i) - h(x_i)]$ where the point set $\{x_i\}$ forms the H-set.

Thus the set of BTA of f by V must be a subset of the set of solutions for the basic equations for any H-set in $N(f)$ and hence must be of the form $h_1 + Q$, where h_1 is a particular solution and Q is some subset of the space of solutions in the homogeneous case. The set Q must be convex due to the convexity of the set of best approximations. The function h_1 can be a BTA to f by V and in this case the zero function is an element of Q.

Consider now the question of uniqueness of a BTA to f by V. First we note that if the matrix $A(M)$ defined by an H-set M has rank n, then the only solution to the set of homogeneous equations

$$(a_1, a_2, \ldots, a_n)\, A(M) = \theta$$

is the trivial solution. Such an H-set must contain a maximal minimal H-set. Using Theorem 2.11, we can now prove the following uniqueness result.

THEOREM 2.12. *If h is a* BTA *to f by V and if $N(f)$ contains a maximal minimal H-set, then h is unique.*

Proof. As was shown above, every set of solutions to basic equations formed from H-sets in $N(f)$ contains the set of BTA to f by V. In the case we are considering, where there is a maximal minimal H-set contained in the set of norm points $N(f)$, the solution set to the basic equations formed from this particular H-set consists of a unique element which must be h. Hence h is the unique BTA to f by V.

This theorem shows that if V satisfies the Haar condition, then every best approximation is unique. Thus Theorem 2.12 includes as a special case the Haar uniqueness theorem.

The converse to the Haar uniqueness theorem can be derived from the more general

THEOREM 2.13. *For every H-set $M = \{x_i\}$ there exists an $f \in C(B)$ such that M is contained in $N(f)$. If* rank $A(M) < n$, *then an f can be found such that there is no unique* BTA *to f by V.*

Proof. Consider an f_0 such that each $f_0(x_i) = e_i$ and $|f(x)| < 1$ otherwise: then the first part of the Theorem follows using the Approximation Theorem: the BTA to f by V is the zero function and $M \subseteq N(f_0)$.

If rank $A(M) < n$, then M lies in some contour $O(h)$ and we can choose h such that $\|h\| = 1$. Then

$$f_1(x) = f_0(x) - \lambda |h(x)| f_0(x)$$

is a required function for suitably small λ.

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